PUTNAM PRACTICE SET 10

PROF. DRAGOS GHIOCA

Problem 1. A sequence $\{x_n\}_{n\geq 0} \subset \mathbb{R}$ is called convex if $2x_n \leq x_{n-1} + x_{n+1}$ for each $n \geq 1$. Let $\{b_n\}_{n\geq 0} \subset \mathbb{R}_+$ be a sequence with the property that for each positive real number a, we have that the sequence $\{a^n b_n\}_{n\geq 0}$ is convex. Then prove that the sequence $\{\log(b_n)\}_{n\geq 0}$ is also convex.

Problem 2. Let $P, Q \in \mathbb{R}[x]$ be monic polynomials satisfying the relation: $P^2(x) = (x^2 - 1)Q^2(x) + 1$ for each $x \in \mathbb{R}$. Prove that $P'(x) = \deg(P) \cdot Q(x)$.

Problem 3. We partition the set $\{1, \dots, 2018\}$ into 6 disjoint subsets $A_1, \dots, A_6$. Prove that there exists some $i \in \{1, \dots, 6\}$ and there exist $x, y, z \in A_i$ (not necessarily distinct) such that $x = y + z$.

Problem 4. Let $\{F_n\}_{n\geq 0}$ be the Fibonacci sequence, i.e.,

$$F_0 = 0,\ F_1 = 1 \text{ and for each } n \geq 2, \text{ we have } F_n = F_{n-1} + F_{n-2}.$$

Prove that each positive integer m can be written uniquely as a sum of non-consecutive (distinct) elements of the Fibonacci sequence, i.e., there exists $\ell \in \mathbb{N}$ and there exist integers

$$1 < i_1 < \cdots < i_\ell \text{ with } i_{j+1} - i_j \geq 2 \text{ for each } j = 1, \dots, \ell - 1$$

such that $m = F_{i_1} + \cdots + F_{i_\ell}$.